Waveguide and Cavity Oscillations in the Presence of Nonlinear Media

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Abstract —This paper deals with the problem of waves in metallic structures containing nonlinear media. Problems of this kind are encountered in the analysis of microwave devices operated at high power levels, or when the constitutive parameters of nonlinear materials are investigated by means of microwave measurements.

The Volterra series are the functional analog of the well-known Taylor series for functions. This mathematical tool is adequate for a description of constitutive relations in dispersive nonlinear media. For practical purposes, we deal with weak nonlinearity, such that the series can be truncated. Weak nonlinearity also denotes the absence of shock waves, such that all spectral components of a wave are phase matched (i.e., propagate with the same phase velocity). The main effect of nonlinearity are the production of harmonics, and the dependence of the dispersion equation on the field amplitudes. These are incorporated into the present model.

The development of the present model involves some heuristic assumptions which facilitate the derivation of an algebraic dispersion equation. Therefore, the range of validity of the present model will have to be determined by experimental results, when these are available.

In waveguides, and cavities in particular, the question of the effect of the geometry and boundary conditions arises, too. It is shown here that nonlinearity induces harmonic modes in rectangular structures. In cylindrical and spherical structures, the geometry affects the budget of harmonics and produces mode coupling.

THE ANALYSIS OF Volterra systems (weakly nonlinear systems with memory) in communication theory was introduced by Wiener in 1942. For relevant literature see Bedrosian and Rice [1], and Bussgang, Eherman, and Graham [2]. The rapid progress in nonlinear optics attracted attention to the corresponding problems in nonlinear wave propagation (e.g., see Caspers [3]). Curiously enough, somewhere along the way the awareness of the fact that we are dealing with Volterra series has been lost (see Caspers [3], Akhmanov and Khokhlov [4], Schubert and Wilhelmi [5], and Censor [6]). The importance of Volterra's original work [7] for the analysis of weakly nonlinear dispersive systems has been recognized recently by Franceschetti and his coworkers [8]–[14] in a series of papers dealing with the theory and application of the Volterra series, and which contain many early references. The systematic application of the Volterra series to problems of wave propagation in weakly nonlinear media is a logical approach, prescribed by the fact that these series are the functional analog of the Taylor series for functions. Recently, this tool has been applied to ray tracing [6], nonlinear wave mechanics [15], solitary wave propagation [16], [17], and nonlinear scattering theory [18]. Recent analytical results and applications are reported by Dalpe, Kent, and Weiner [19]. For completeness, the fundamental theory used here will be summarized.

Presently, propagation in weakly nonlinear media is applied to problems of guided waves and cavity oscillations. The motivation for this study is threefold. Firstly, classical results for rectangular and circular waveguides, and similar cavity configurations, are relatively easy to extend to the case of nonlinear media. At the time when relevant experimental results will be available, the present results will serve to check the theory used here, which involves some yet unjustified heuristic assumptions. Secondly, for some devices used to date, linearized theory only is used for low power levels. The present method provides nonlinear corrections. And thirdly, just as microwave devices are presently used to investigate the linear properties of various materials, the present theory provides the basis for diagnosing the nonlinear properties. This kind of problem also seems to be of interest in connection with nonlinear acoustics [20].

Starting with a summary of the relevant theory, the propagation of plane periodic waves in unbounded media is considered. The question of the validity of superposition of nonphase-matched waves is discussed and general solutions obtained. The results are applied to canonical waveguide and cavity problems. The case of rectangular waveguides and cavities is conceptually and mathematically simpler. It shows the production of harmonics and the associated modes, as well as the effect of the amplitude on the dispersion equation. The problems of circular, cylindrical, and spherical systems shows what phenomena should be expected (according to the present theory) when curved boundaries are present. It is shown that curved boundaries might suppress harmonic production in certain cases. Also, nonlinearity acts as a mode-coupling mechanism, as shown below. These are interesting phenomena which will have to be tested experimentally.

II. General Theory of Self-Interaction

The problem is considered in the frame of electrodynamics in sourceless domains, governed by Maxwell's equations

$$\nabla \times \boldsymbol{E} + \partial \boldsymbol{B}/\partial t = 0 \quad \nabla \times \boldsymbol{H} - \partial \boldsymbol{D}/\partial t = 0$$
$$\nabla \cdot \boldsymbol{D} = 0 \quad \nabla \cdot \boldsymbol{B} = 0. \tag{1}$$

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For reasons explained below, periodic (as opposed to harmonic) plane-wave solutions are considered

$$E_i = \sum_{q=-\infty}^{\infty} E_{qi} e^{iq\theta}, \qquad \theta = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t \tag{2}$$

and similar expression for B_i, H_i, D_i, where $i=1,2,3$, denotes Cartesian components, q is the order of the harmonic, θ is the phase of the plane wave, E_{qi} the amplitude of the i component of the qth harmonic, $\boldsymbol{k}$ is the propagation vector, and $\boldsymbol{x}$ the position vector. Substitution of (2) etc. in (1) yields

$$\boldsymbol{k}\times\boldsymbol{E}_q - \omega\boldsymbol{B}_q = 0 \quad \boldsymbol{k}\times\boldsymbol{H}_q + \omega\boldsymbol{D}_q = 0 \tag{3}$$

and $\nabla\cdot\boldsymbol{D}_q = 0$, $\nabla\cdot\boldsymbol{B}_q = 0$ are identically satisfied. Before going on, a few observations are in order. The periodic solution, as in (2), has been stipulated in order to include harmonic production due to distortion by the nonlinear medium. The amplitudes E_{qi}, D_{qi}, H_{qi}, and B_{qi} are not arbitrary but determined by (3) and the complicated constitutive relations to be introduced below. It is noted that solutions of the kind in (2) already assume phase matching, i.e., all harmonics have identical phase velocities $\omega/|\boldsymbol{k}|$. We are dealing with weakly nonlinear media, in which the creation of shock waves is excluded. This is adequately described by (2), because shock formation requires that different spectral components propagate with different phase velocities. Phase matching also means that harmonics are produced in a coherent manner, i.e., local nonlinear interactions, although they might be weak, produce waves which interfere constructively, producing significant amplitudes of harmonic waves.

In general, nonlinear constitutive relations are given by $\boldsymbol{D}(\boldsymbol{E})$, $\boldsymbol{B}(\boldsymbol{H})$, but for weak nonlinearity, a hierarchy is assumed

$$\boldsymbol{D} = \boldsymbol{D}^{(1)} + \boldsymbol{D}^{(2)} + \cdots + \boldsymbol{D}^{(n)} + \cdots \tag{4}$$

and similarly for $\boldsymbol{B}$. In (4), it is assumed that the leading terms are predominant, and for practical purposes the sum (4) can be truncated. The term $\boldsymbol{D}^{(n)}$ in (4) is defined as the nth term of a Volterra series

$$D_i^{(n)}(\boldsymbol{x},t) = \int d^3x_1\,dt_1 \cdots \int d^3x_n\,dt_n\left[\epsilon_{i,j\cdots\nu}^{(n)}(\boldsymbol{x}_1,t_1,\cdots \boldsymbol{x}_n,t_n)E_j(\boldsymbol{x}-\boldsymbol{x}_1,t-t_1)\cdots E_\nu(\boldsymbol{x}-\boldsymbol{x}_n,t-t_n)\right] \tag{5}$$

where indices $i, j,\cdots,\nu$ denote Cartesian components and $\epsilon_{i,j\cdots\nu}^{(n)}$ is a tensor (Einstein's summation convention is assumed). Thus, for $n=1$, we obtain the linear case, which on substitution of (2) yields

$$D_i^{(1)}(q\boldsymbol{k},q\omega) = \epsilon_{i,j}^{(1)}(q\boldsymbol{k},q\omega)E_j(q\boldsymbol{k},q\omega) \tag{6}$$

for the qth harmonic, where $\epsilon_{i,j}^{(1)}(q\boldsymbol{k},q\omega)$ is the four-dimensional transform according to

$$(2\pi)^{-4}\int d^3x_1 dt_1 \epsilon_{i,j}^{(1)}(\boldsymbol{x}_1,t_1)e^{-iq\theta(x_1,t_1)} \tag{7}$$

between infinite limits of integration. The parameters $\epsilon_{i,j}^{(n)}$ of (5) are constants characterizing the system. The first nonlinear term $D^{(2)}$ upon substitution of (2) yields

$$D_i^{(2)}(\boldsymbol{x},t) = \sum_q\sum_{q'} e^{i(q+q')\theta(x,t)}\epsilon_{ijk}^{(2)}(q\boldsymbol{k},q\omega,q'\boldsymbol{k},q'\omega)E_{qj}E_{q'k} \tag{8}$$

and $D^{(n)}(\boldsymbol{x},t)$ involves n summations [6]. If periodic $D_i^{(2)} = \sum_\gamma D_{\gamma i}e^{i\gamma\theta}$ is stipulated and substituted in (8), and terms satisfying $\gamma = q+q'$ are regrouped, then we have

$$D_{\gamma i}^{(2)} = \bar{\epsilon}_{ijk}^{(2)}(\gamma\boldsymbol{k},\gamma\omega)E_{\gamma j}E_{\gamma k} \tag{9}$$

where we define

$$\bar{\epsilon}_{ijk}^{(2)}(\gamma\boldsymbol{k},\gamma\omega) = \sum_{q,q'=\gamma-q}\epsilon_{ijk}^{(2)}\frac{E_{qj}E_{q'k}}{E_{\gamma j}E_{\gamma k}}. \tag{10}$$

Originally $\epsilon^{(n)}$ in (5) (hence, also the $4\times n$-dimensional transform in (8)) are defined as parameters of the medium, independent of field amplitudes. Consequently, (10) etc. display $\bar{\epsilon}^{(n)}$ as amplitude dependent; hence, the notation (9) seems to be useless. What is the point of defining characteristic parameters if it turns out that they depend on the variable fields? What we are trying to achieve, and this is a crucial step which has to be tested experimentally, is the following. A plane wave injected into a nonlinear medium will be distorted, i.e., it will undergo self interaction, until a periodic wave is present satisfying Maxwell's equations (3) and the constitutive relations (4) and (5), and the corresponding relations $\boldsymbol{B}(\boldsymbol{H})$. The budget of amplitudes of various harmonics depends on the original excitation and the properties of the medium. In other words, if a periodic plane wave is injected into the medium with exactly the right amplitudes and relative phases of harmonics, this wave will propagate in the medium without modification. The assumption implied in (9) is that the ratios of amplitudes are insensitive to incremental variation, i.e., if all the amplitudes are increased by a small factor, the ratio in (10) becomes

$$\frac{E_{qj}E_{q'k}}{E_{\gamma j}E_{\gamma k}}\left(1+\frac{\Delta E_{qj}}{E_{qj}}+\frac{\Delta E_{q'k}}{E_{q'k}}-\frac{\Delta E_{\gamma j}}{E_{\gamma j}}-\frac{\Delta E_{\gamma k}}{E_{\gamma k}}\right) \tag{11}$$

and ideally the expression in parentheses in (11) equals 1. As long as the increments in (11) are small enough to justify this approximation, (9) is valid as an approximation. According to (4) and (9), we have for each harmonic q

$$D_i = \bar{\epsilon}_{ij}^{(1)}E_j + \bar{\epsilon}_{ijk}^{(2)}E_jE_k + \cdots + \bar{\epsilon}_{ij\cdots\nu}^{(n)}(E_j\cdots E_\nu) + \cdots \tag{12}$$

and if considered as an expansion of $\boldsymbol{D}(\boldsymbol{E})$ about $\boldsymbol{E}=0$, then $\bar{\epsilon}_{ij\cdots\nu}^{(n)}$ are the values of the derivatives in a Taylor expansion

$$\bar{\epsilon}_{ij\cdots\nu}^{(n)} = \frac{1}{n!}\frac{\partial^n D_i}{\partial E_j\cdots\partial E_\nu}. \tag{13}$$

Many studies heuristically start with (12), which is not as systematic as the present approach. The systematic approach through the tool of Volterra's series clearly displays $\bar{\epsilon}^{(n)}$ as only approximately constant, as opposed to $\epsilon^{(n)}$.

At this point, (9) etc. is substituted in (3), and for $q=1$, this yields

$$(\boldsymbol{k}\times\boldsymbol{E})_i - \omega\bar{\mu}^{(1)}_{ij}H_j - \omega\bar{\mu}^{(2)}_{ijk}H_jH_k - \cdots = 0$$
$$(\boldsymbol{k}\times\boldsymbol{H})_i + \omega\bar{\epsilon}^{(1)}_{ij}E_j + \omega\bar{\epsilon}^{(2)}_{ijk}E_jE_k + \cdots = 0 \quad (14)$$

where the index $q=1$ has been suppressed, and similar expressions exist for higher harmonic waves. The six homogeneous, algebraic, nonlinear scalar equations (14) define the physical problem at hand. The system (14) can be used to deliver an amplitude dependent dispersion equation, as explained previously [6]. To clarify this, we shall handle the situation in a somewhat primitive way. The second equation (14) can be manipulated to derive expressions for H_1, H_2, H_3 in terms of E_1, E_2, E_3, and these are substituted in the first line of (14), resulting in three scalar equations on E_1, E_2, E_3. In each of these equations, there are terms that do not involve E_1, say; hence, the set of equations may be written in the form

$$E_1f_1(E_1,E_2,E_3) = g_1(E_2,E_3)$$
$$E_1f_2(E_1,E_2,E_3) = g_2(E_2,E_3)$$
$$E_1f_3(E_1,E_2,E_3) = g_3(E_2,E_3) \quad (15)$$

where f and g are arbitrary functions of the arguments, and in some degenerate cases not all the arguments will be present. By elimination of the factors E_1 on the left, in (15), we obtain two scalar equations which have the general form

$$E_2h_1(E_1,E_2,E_3) = l_1(E_1,E_3)$$
$$E_2h_2(E_1,E_2,E_3) = l_2(E_1,E_3) \quad (16)$$

and finally, by dividing the equations in (16), one equation of the form

$$E_3u(E_1,E_2,E_3) = 0. \quad (17)$$

Hence, for the nontrivial solution $E_3 \neq 0$ in (17) prescribes a dispersion relation $u=0$, which involves $\boldsymbol{k}, \omega$, and field amplitudes. This procedure is equivalent to writing (14) in matrix form $G_r = F_{rs}A_s = 0$, $r,s = 1,\cdots,6$, where $\boldsymbol{A} = (A_s) = (E_1, E_2, E_3, H_1, H_2, H_3)$ is a six component vector, and by imposing the condition of solubility $\det(F_{rs}) = 0$, the dispersion equation

$$F(\boldsymbol{k},\omega,\boldsymbol{A}) = 0 \quad (18)$$

is obtained. The fact that (18) involves amplitudes is a characteristic feature of the nonlinear problem. The corresponding equations (14) and (18) for higher harmonics are not independent systems of equations, because (14) and (18) already establish a relation between $\boldsymbol{k}$ and ω. Hence, the equations for higher harmonics can only serve to determine the (complex) amplitudes of the harmonic waves. The details are not very important to the main line of our subject.

At this point, we have sufficiently summarized the general theory in order to discuss metallic guides and resonators. We begin, in the next section, with the relatively simple problem of rectangular guides and cavities.

III. Rectangular Waveguides and Resonators

In order to demonstrate the feasibility of using the above theory for rectangular waveguides and cavities, a simple isotropic constitutive relation is used. This class of problems is still general enough to display typical aspects of the nonlinear class of problems. Accordingly, we define a dielectric medium with scalar constant μ and

$$D_i = \bar{\epsilon}^{(1)}_{ij}E_j + \bar{\epsilon}^{(2)}_{ijk}E_jE_k + \cdots$$
$$= \epsilon^{(1)}E_i + \epsilon^{(2)}(E_i)^2 + \epsilon^{(3)}(E_i)^3 + \cdots \quad (19)$$

where for expressions containing scalar $\epsilon^{(n)}$, the summation convention is inapplicable. This means that $\bar{\epsilon}^{(1)}_{ij}$ is diagonalized by multiplying by a Kroenecker δ_{ij}, and the diagonal elements made identical, and a corresponding treatment for the higher order tensors. Equation (19) is compacted in the form

$$\boldsymbol{D} = \epsilon_{\text{eff}}(\boldsymbol{E})\boldsymbol{E} \quad (20)$$

where it is understood that the field $\boldsymbol{E}$ in $\epsilon_{\text{eff}}(\boldsymbol{E})$ is the field E_i related to D_i. Manipulating Maxwell's equations (3) yields $(q=1)$

$$\boldsymbol{k}\times\boldsymbol{k}\times\boldsymbol{E} + \omega^2\mu\epsilon_{\text{eff}}(\boldsymbol{E})\boldsymbol{E} = 0.$$

This medium admits transversal waves for which the wave equation becomes

$$\left[k^2 - \omega^2\mu\epsilon_{\text{eff}}(\boldsymbol{E})\right]\boldsymbol{E} = 0. \quad (21)$$

Hence, the expression in brackets is the dispersion equation (18).

The main difficulty in proceeding to analyze the present problem is that, unlike the linear case, the representation of the total field as a superposition of plane waves requires justification. In general, superposition is not valid in nonlinear media. However, in weakly nonlinear systems as discussed here, it appears plausible to assume that only phase-matched nonlinear induced harmonics will be produced with significant efficiency. This implies that in isotropic media as considered here, the interaction of noncolinear waves will be negligible. This heuristic assumption, still requiring experimental support, salvages the linear method of superposition to the extent that new solutions may be constructed from sums of plane waves which are not phase matched.

Accordingly, we take the formulas for rectangular waveguides, e.g., as given by Collin [21], recast them in terms of plane waves, and replace $\boldsymbol{k}, \omega$ with $q\boldsymbol{k}, q\omega$, respectively, to obtain the harmonics. Thus, the fields are given by

Field	TE	TM	
H_z	C_xC_ye	0	
E_z	0	S_xS_ye	
E_x	$Z_{h,nm}H_y$	AC_xS_ye	(22)
E_y	$-Z_{h,nm}H_x$	BS_xC_ye	
H_x	$-AS_xC_ye$	$-E_y/Z_{e,nm}$	
H_y	$-BC_xS_ye$	$E_x/Z_{e,nm}$	

where in the above shorthand notation $C_x = \cos(qn\pi x/a)$, and S_x implies sine of the same argument, and $C_y = \cos(qm\pi y/b)$, and S_y follows; a and b are the dimensions of the guide in the x and y directions, respectively, q is the number of the harmonics, e denotes $e^{iq\beta_{nm}z - iq\omega t}$, where $q\beta_{nm}$ is the pertinent component of $q\mathbf{k}$ in the z-direction, A denotes $i\beta_{nm}n\pi/ak^2_{c,nm}$ and for B replace n/a with m/b, $k^2_{cnm} = (n\pi/a)^2 + (m\pi/b)^2$, and $\beta_{nm} = k^2 - k^2_{c,nm}$. The impedances are given by $Z_{h,mn} = kZ_0/\beta_{nm}$, $Z_{e,nm} = \beta_{nm}Z_0/k$, and $Z_0 = \omega\mu/k$, and, for the present case in particular, $Z_0 = \mu/\epsilon_{\text{eff}}(\mathbf{E})$, displaying the dependence on the amplitude of the first harmonic.

The following characteristics of the nonlinear problem are therefore apparent from (22): since nonlinearity produces harmonics, the fields of the fundamental mode n, m appear together with harmonic modes (i.e., at frequencies $q\omega$) qn, qm. Since phase matching is built into the model from the outset, all these modes propagate with identical phase and group velocities. Any attempt to change the balance of harmonics, e.g., to filter out some harmonic or to add a wave at a certain harmonic frequency, results in a change of the parameters $\bar{\epsilon}^{(n)}$ in (10); hence, there is a reshuffling of the whole spectral content of the wave. To deal with $\bar{\epsilon}^{(n)}$, which are susceptible to these changes, is the price we have to pay for simplifying the nonlinear model from (8) to (9). The second main result is the dependence of parameters on $\epsilon_{\text{eff}}(\mathbf{E})$; hence, the amplitude of the excitation affects the wavelength, impedances, etc. It is therefore possible to regulate certain parameters, e.g., cutoff frequencies, by changing the amplitude. This property is usually referred to in nonlinear optics as nonlinearity-induced transparency.

Waveguides containing nonlinear media as part of their structure can be analyzed by the above formalism. Conversely, nonlinear media can be analyzed by inserting them into waveguide systems.

The extension of (22) to rectangular resonators is quite straightforward. To the waves in (22), add (with proper signs) backward propagating waves such that metallic boundary conditions will be satisfied at $z = 0, d$. Normalized formulas have the form

Field	TE	TM
H_z	$C_xC_yS_ze$	0
E_z	0	$S_xS_yC_ze$
E_x	$-Z_{h,nm}BC_xS_yS_ze$	$iAC_xS_yS_ze$
E_y	$Z_{h,nm}AS_xC_yS_ze$	$iBS_xC_yS_ze$
H_x	$iAS_xC_yC_ze$	$-BS_xC_yC_ze/Z_{e,nm}$
H_y	$iBC_xS_yC_ze$	$AC_xS_yC_ze/Z_{e,nm}$

(23)

where e stands for $e^{-iq\omega t}$ and $S_z = \sin(ql\pi z/d)$, $C_z = \cos(ql\pi z/d)$, $l = 1, 2, 3, \cdots$. Each of the plane waves contained in (23), which can be explicitly obtained by recasting the sin and cos as functions in exponentials, has $\mathbf{k}$ vector components $k_x = \pm n\pi/a$, $k_y = \pm m\pi/b$, $k_z = \pm l\pi/d$ with the appropriate sign. Hence, we are dealing with eight non-codirectional waves that can be superposed, since they are not phase matched. Hence, the dispersion relation (18) can be written as

$$F\left(\frac{n\pi}{a}, \frac{m\pi}{b}, \frac{l\pi}{d}, \omega, A\right) = 0 \tag{24}$$

which is amplitude dependent. Consequently, the resonance frequency is also amplitude dependent. This is the key to measuring

$$\epsilon_{\text{eff}}(\mathbf{E}) = k^2/(\mu\omega^2). \tag{25}$$

For small amplitudes, the nonlinear effect is negligible. As the power increases, the nonlinear correction terms will start to play a role and can be computed from the change in the resonance frequency.

IV. Circular Cylindrical Structures

It will be shown that this class of canonical problems is not merely a complicated mathematical extension of the rectangular case, in which Bessel functions replace the trigonometric expressions. In fact, if we adopt the general theory given above, then we find that curved metallic structures, as opposed to rectangular geometries, will usually suppress phase matching, and consequently also suppress coherent nonlinear interaction.

The general treatment of linear vector waves is given by Stratton [22], who cites original work by Hansen. For completeness, and since the subject is mathematically more complicated than the rectangular case, the general theory is summarized. The general expressions for nonsingular fields in cylindrical coordinates [22, see p. 361] is

Field	TM	TE
E_r	$ih\Sigma a_n \partial\psi_n/\partial r$	$-\mu\omega/r\Sigma nb_n\psi_n$
E_ϕ	$(-h/r)\Sigma na_n\psi_n$	$-i\omega\mu\Sigma b_n\partial\psi_n/\partial r$
E_z	$\lambda^2\Sigma a_n\psi_n$	0
H_r	$(k^2/\mu\omega r)\Sigma na_n\psi_n$	$ih\Sigma b_n\partial\psi_n/\partial r$
H_ϕ	$(ik^2/\mu\omega)\Sigma a_n\partial\psi_n/\partial r$	$(-h/r)\Sigma nb_n\psi_n$
H_z	0	$\lambda^2\Sigma b_n\psi_n$

(26)

where

$$\psi_n = e^{in\phi}J_n(\lambda r)e^{ihz - i\omega t}. \tag{27}$$

In denoting the nonsingular Bessel functions and $\lambda^2 = k^2 - h^2$, the summation extends on $-\infty < n < \infty$ and a_n, b_n are coefficients. The structure (26) and (27) can be split into even and odd parts by defining

$$\psi_o^{e_n} = \begin{Bmatrix} \cos n\phi \\ \sin n\phi \end{Bmatrix} J_n(\lambda r)e^{ihz - i\omega t}. \tag{28}$$

In order to express (26) in terms of (28), note that *in* in (26) corresponds to $\partial/\partial\phi$, and apply this operator to (28) (see Stratton [22, p. 395]). The orthogonal vector wave functions $\mathbf{M}$ and $\mathbf{N}$ are defined in Stratton [22, p. 392 ff.]

$$\mathbf{M}_n = (\nabla\psi_n)\times\hat{z} \quad \nabla\times\mathbf{M}_n = k\mathbf{N}_n$$
$$\nabla\times\mathbf{N}_n = k\mathbf{M}_n. \tag{29}$$

The TE $\mathbf{E}$ field in (26) is recognized as $i\omega\mu\Sigma b_n\mathbf{M}_n$, and the mate $\mathbf{H}$ field corresponds to $k\Sigma b_n\mathbf{N}_n$; the TM $\mathbf{H}$ field is recognized as $(-ik^2/\mu\epsilon)\Sigma a_n\mathbf{M}_n$ and the mate $\mathbf{E}$ field is given by $(k\omega/\epsilon)\Sigma a_n N_n$. Relevant to our subject is the representation of $\boldsymbol{M}$ and $\boldsymbol{N}$ in terms of sums (integrals) of plane waves

$$\boldsymbol{M}_n = \frac{i^{-n}}{2\pi}\int_0^{2\pi}(i\boldsymbol{k}\times\hat{z})\,e^{i\lambda r\cos(\beta-\phi)+in\beta+ihz-i\omega t}\,d\beta$$

$$\boldsymbol{N}_n = \frac{i^{-n}}{2\pi}\int_0^{2\pi}(k\hat{z}-h\hat{\boldsymbol{k}})\,e^{i\lambda r\cos(\beta-\phi)+in\beta+ihz-i\omega t}\,d\beta. \tag{30}$$

(See Stratton [22, pp. 396, 397]).

Thus far the linear problem has been summarized. Strictly speaking, the nonlinear problem does not admit superposition of plane waves. However, making the same assumption as before, that it is legitimate to superpose waves which are not phase matched, structures like (30) are admissible. But exactly this argumentation also leads to the prediction that the significance of nonlinear interaction will be very small, since for each amplitude $i\boldsymbol{k}\times ze^{in\beta}$ or $(k\hat{z}-h\hat{\boldsymbol{k}})e^{in\beta}$, the nonlinear dispersion equation (18) yields a different value for $\boldsymbol{k}$, and in the absence of some concerted effort of all the plane waves in the integrand (30), the effect of nonlinearity will be negligible. One way of dealing with this difficulty is to assume the presence of many modes, and let the amplitudes affect the dispersion equation such that a combined coherent effect will emerge. Thus, we define the $\boldsymbol{E}$ field corresponding to (30) as

$$\frac{1}{2\pi}\int_0^{2\pi}d\beta e^{i\lambda r\cos(\beta-\phi)+ihz-i\omega t}\boldsymbol{E}(\beta) \tag{31}$$

where $\boldsymbol{E}(\beta)=\Sigma_n\boldsymbol{E}_n(\beta)e^{in\beta}$ stands for a sum of amplitudes of various modes. Assuming again the isotropic medium as in (19) and (21) prescribes

$$\frac{1}{2\pi}\int_0^{2\pi}d\beta e^{i\lambda r\cos(\beta-\phi)+ihz-i\omega t}\{k^2\boldsymbol{E}(\beta)-\omega^2\mu\epsilon^{(1)}\boldsymbol{E}(\beta) - \omega^2\mu\epsilon^{(2)}E^2(\beta)\hat{E}(\beta)-\omega^2\mu\epsilon^{(3)}E^3(\beta)\hat{E}(\beta)-\cdots\}=0. \tag{32}$$

What we are trying to do now is to find conditions for the vanishing of the expression in braces in (32) for all β. For simplicity, let us assume first that only $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are nonzero. The condition for the vanishing of the braces in (32) for all β prescribes

$$(k^2-\omega^2\mu\epsilon^{(1)})\sum_n b_n e^{in\beta}-\omega^2\mu\epsilon^{(2)}\sum_{n'}\sum_{n''}b_{n'}b_{n''}e^{i(n'+n'')\beta}=0 \tag{33}$$

for the TE field, for example. Hence, using the orthogonality of $e^{im\beta}$, (33) yields

$$(k^2-\omega^2\mu\epsilon^{(1)})b_n-\omega^2\mu\epsilon^{(2)}\sum_{n'+n''=n}b_{n'}b_{n''}=0. \tag{34}$$

For each n, an equation of the type (34) is obtained, and since all factors except k are given beforehand, each equation yields a value k_n. Since λ is determined by boundary conditions, (34) in general means that different k_n are associated with different h_n; hence, our assumption (31) (with one h applying to all modes n) is invalid. Still, our argument may be applied as an approximation. If, for $\epsilon^{(2)}=0$, a value h_0 is computed, then the nonlinear effect is

$$\Delta h_n = h_n - h_0. \tag{35}$$

If Δh_n is small compared to h_0, then the phase mismatch $e^{i\Delta h_n z}$ will be negligible for some range of z, for which h in (31) can be replaced by h_0. Naturally, this means that for high-order modes, where h is getting smaller compared to λ, the approximation is increasingly improved. This is to be expected, because for higher modes, corresponding to larger arguments in the Bessel functions, the asymptotic representation of the Bessel functions constitutes a good approximation. This is tantamount to saying that as λ increases, the waves become more and more similar to plane waves. The above argument shows the interaction of modes, which also means that if one mode n is injected into the system, then powers of $e^{in\beta}$ will produce higher modes, whose amplitudes contain the nonlinear $\epsilon^{(n)}$ as factors. Subject to the restrictions mentioned above, the field in circular cylindrical waveguides can be represented by (27) and (28) with the proper modifications, i.e., h, k, are replaced by h_n, k_n, obtained from (34) and the boundary condition on λ, according to $k_n^2=\lambda^2+h_n^2$. Similarly to the previous case of rectangular cavities, here too the resonance frequency of a cylindrical cavity will depend on the amplitude of the injected signal. For this case, h is determined by the length z of the cavity, and the only degree of freedom is offered by ω. If only one mode is present, which we can represent by $\pm n$, $|b_n|=|b_{-n}|$, then the same argument that led to (34) now yields, for $\epsilon^{(3)}$, a relation of the form

$$(k^2-\omega_n^2\mu\epsilon^{(1)})b_n-\omega_n^2\mu\epsilon^{(3)}3b_n^3=0 \tag{36}$$

which is somewhat oversimplified, but shows the dependence of ω_n on the amplitude, represented here by b_n.

Thus far only the fundamental frequency has been considered. Inasmuch as we were able to recast the fields in plane-wave integrals, it is clear that for frequency $q\omega$ we will not have qk (i.e., $q\lambda$ and qh) to maintain the phase-matching requirement. However, if the boundary conditions prescribe $J_n(\lambda a)=0$ or $(d/da)J_n(\lambda a)=0$, for TM and TE modes, respectively, for a guide of radius $r=a$, then in general $J_n(q\lambda a)$, $(d/da)J_n(q\lambda a)$ will not vanish due to the fact that the zeros of the functions are not evenly spaced. This means that harmonic production is suppressed in cylindrical waveguides and cavities. For high-order modes, the zeros become increasingly evenly spaced, because the waves resemble more and more plane waves. Hence, for large λa, the harmonic waves will be present.

V. Spherical Structures

At this stage, where a lot of experimentation is needed to check the fundamentals of the theory, there is no point in bringing in all the heavy machinery for the scalar and vector spherical wave functions. These are comprehensively covered by Stratton [22, see ch. 7]. The ideas are identical, and therefore we expect the same conclusions. The representation of spherical vector waves in terms of plane-wave integrals [22, pp. 416, 417], and the application of dispersion equations similar to (33) and (34) will lead to the results of k_{nm} for mode n, m which constitutes an approximate solution, increasingly improving as n increases (on account of the spherical Bessel functions behavior for large arguments). The conclusions for the behavior of harmonics follows the same lines.

V. Summary and Conclusions

The problem of nonlinear wave propagation is extremely complicated, physically and mathematically. The present study concentrates on weak nonlinear effects which provide the correction terms for the leading linear results. This is described mathematically by a model based on the Volterra series, and plane-wave dispersion relations are obtained by assuming periodic solutions. This theory is briefly recapitulated. Applications to rectangular waveguides and cavities are given. This problem is easy because the linear fields are given as combinations of a few (at most eight, for the fully developed case of a rectangular cavity) plane waves. Once the stipulation is made that nonphase-matched waves do not interact, the extension to the nonlinear case is straightforward. Results are given, and practical aspects of analyzing nonlinear devices, or measuring the properties of nonlinear media, are discussed.

The presence of curved metallic boundaries is shown to suppress nonlinear interaction and harmonic production. This effect is increasingly pronounced as the waves depart more and more from plane waves, i.e., when the curvature of wavefronts increases. Cylindrical waves are considered in some detail, the treatment for spherical structures is only delineated, but the above conclusions seem to be valid in general.

There are many heuristic assumptions in the basic theory, and experimental data is necessary to check its validity.

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